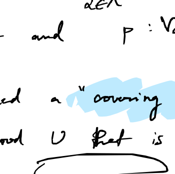
4/16

COVERING SPACES

Def: Let $p: E \to B$ be a continuous, surjective map. The open set $U \subset B$ is "evenly covered" by p if $p^{-1}(U) = \bigcup_{\alpha \in \Lambda} V_\alpha$ where V_α's are pairwise disjoint and $p: V_\alpha \to U$ is a homeomorphism.

p is called a "covering map" if every $y \in B$ has a neighborhood U that is evenly covered.



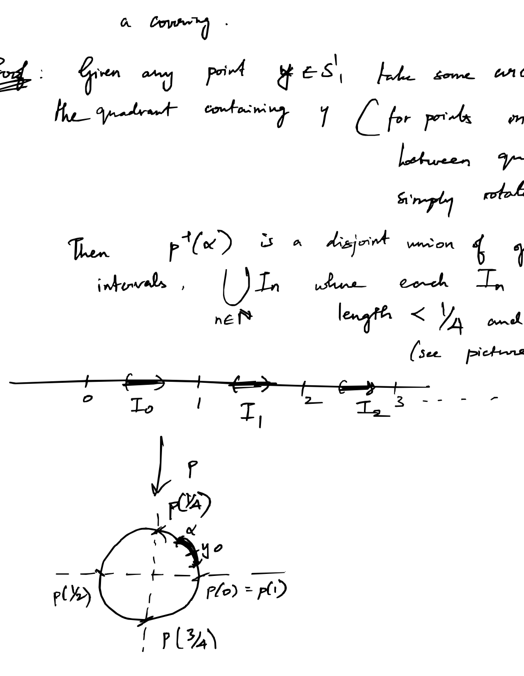
If $p: E \to B$ is a covering map, we call E a "covering space of B".

Eg: The identity map of any space is a covering map.

Theorem: The map $p: \mathbb{R} \to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering.

Proof: Given any point $y \in S^1$, take some arc α strictly in the quadrant containing y (for points on the boundary between quadrants, simply rotate).

Then $p^{-1}(\alpha)$ is a disjoint union of open intervals, $\bigcup_{n \in \mathbb{N}} I_n$ where each I_n has length $< 1/4$ and $I_{n+1} = I_n + 1$ (see picture).



Another Example:
$p: S^1 \to S^1$ given by $p(z) = z^2$ is a covering map.

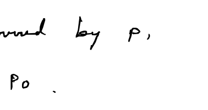
Proof: Exercise.

Theorem: Let $p: E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then $p_0: E_0 \to B_0$ obtained by restricting p is a covering map.

Proof: any $U \subset B_0$ open is of the form $\tilde{U} \cap B_0$ for some $\tilde{U}$ open in B.

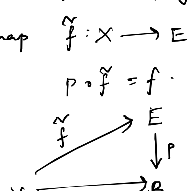
If $y \in B_0$ and $\tilde{U}$ is evenly covered by p, then U is evenly covered by p_0.

Theorem: If $p: E \to B$ and $p': E' \to B'$ are covering maps, then $(p, p'): E \times E' \to B \times B'$ is a covering map.

Eg: For the covering $p: \mathbb{R} \to S^1$ discussed above, $p \times p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map. The space $S^1 \times S^1$ is called a "torus".
(It can be visualized as a donut )

Def: Let $p: E \to B$ be a map. Given any cts map $f: X \to B$, a "lifting of f through p" is a cts map $\tilde{f}: X \to E$ s.t.

$p \circ \tilde{f} = f$.



Lemma: Let $p: E \to B$ be a covering map and $p(e_0) = b_0$. Any path f in B beginning at b_0 has a unique lifting to a path in E beginning at e_0.

Proof: For every $t \in [0,1]$, the point $b_t = f(t)$ has an evenly covered nbhood U_t for p.
The collection $\{U_t\}_{t \in [0,1]}$ is an open cover of the compact set $f([0,1])$ (the image of the path). Therefore it has a finite subcover $U_{t_1}, \dots, U_{t_k}$.

Note: $\exists$ a subdivision $0 = a_0 < a_1 < \dots < a_{k-1} = 1$ of $[0,1]$ so that $f|_{[a_i, a_{i+1}]}$ lies in $U_{t_{i+1}}$ $\forall i$.

Define $\tilde{f}: [0,1] \to E$ as follows:
$\because U_{t_1}$ is evenly covered, $\exists V_{t_1} \ni e_0$ open s.t. $p: V_{t_1} \to U_{t_1}$ is a homeomorphism.

set $\tilde{f}|_{[a_0, a_1]} = (p|_{V_{t_1}})^{-1} \circ f|_{[a_0, a_1]}$.

Now note that $\tilde{f}(a_1)$ is s.t.
$p \circ \tilde{f}(a_1) = f(a_1) \in U_{t_2}$.

$\therefore \exists V_{t_2} \ni \tilde{f}(a_1)$ s.t. $p: V_{t_2} \to U_{t_2}$ is a homeomorphism. So we let

$\tilde{f}|_{[a_1, a_2]} = (p|_{V_{t_2}})^{-1} \circ f|_{[a_1, a_2]}$.

Continuing in this way, we can define each $\tilde{f}|_{[a_i, a_{i+1}]}$, and in this way, a path lifting $\tilde{f}$ of f through p that starts at e_0.

Uniqueness: Exercise.

Lemma: Let $p: E \to B$ be a covering map, let $p(e_0) = b_0$. Let $F: [0,1] \times [0,1] \to B$ be continuous, with $F(0,0) = b_0$. There exists a unique lifting of F to a cts map

$\tilde{F}: [0,1] \times [0,1] \to E$ s.t.
$\tilde{F}(0,0) = e_0$. If F is a path homotopy, so is $\tilde{F}$.

Proof: Very similar to the above, but instead of taking a subdivision of $[0,1]$, we instead take a subdivision of $[0,1] \times [0,1]$ into small rectangles $R_{i,j} = [a_i, a_{i+1}] \times [b_j, b_{j+1}]$ s.t. $F|_{R_{i,j}}$ lies in an evenly covered neighborhood in B.

As before, we use this to define $\tilde{F}|_{R_{i,j}}$.

Theorem: Let $p: E \to B$ be a covering map, let $p(e_0) = b_0$. Let f, g be two paths in B from b_0 to b_1, let $\tilde{f}, \tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f, g are both path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Proof: Say $f \simeq_p g$ via F. Let $\tilde{F}$ be a lifting of F to E so that $\tilde{F}(0,0) = e_0$.
By the preceding lemma, $\tilde{F}$ is a path homotopy, so that $\tilde{F}([0,1] \times 0) = \{e_0\}$ and $\tilde{F}([0,1] \times 1)$ is a one-point set $\{e_1\}$.

$\tilde{F}|_{0 \times [0,1]}$ is a path on E beginning at e_0 that lifts f. By uniqueness of path liftings, we have $\tilde{F}|_{0 \times [0,1]} = \tilde{f}$.
Similarly, $\tilde{F}|_{1 \times [0,1]} = \tilde{g}$.

Def: Let $p: E \to B$ be a covering map and $b_0 \in B$. Choose $e_0 \in p^{-1}(b_0)$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path beginning at e_0. Let $\Phi([f])$ denote the end point of $\tilde{f}(1)$ of $\tilde{f}$. Then Φ is a well-defined set map

$\Phi: \pi_1(B, b_0) \to p^{-1}(b_0)$.

Φ is called the "lifting correspondence" derived from p. Φ depends on e_0.

Theorem: Let $p: E \to B$ be a covering map, and $p(e_0) = b_0$. If E is path connected, then $\Phi: \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective.
If E is simply connected, it is bijective.

Proof: If E is path connected, then given $e_1 \in p^{-1}(b_0)$, $\exists$ a path $\tilde{f}$ from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B based at b_0, and we have $\Phi([f]) = e_1$. Thus Φ is surjective.

If E is simply connected, let $[f], [g]$ be s.t. $\Phi([f]) = \Phi([g])$. In other words, $\tilde{f}(1) = \tilde{g}(1)$. Then $\tilde{f} * \overline{\tilde{g}}$ is a loop based at e_0. $\because E$ is simply connected, $\exists$ a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g, i.e., $[f] = [g]$. Thus Φ is injective.

Theorem: $\pi_1(S^1, (1,0)) \cong (\mathbb{Z}, +)$.

Proof: Let $p: \mathbb{R} \to S^1$ be the covering map $x \mapsto (\cos 2\pi x, \sin 2\pi x)$.

$p^{-1}((1,0)) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected
$\Phi: \pi_1(S^1, (1,0)) \to \mathbb{Z}$ is bijective.

Claim: Φ is a homomorphism.

Proof: Given $[f], [g]$ in $\pi_1(B, b_0)$, let $\tilde{f}, \tilde{g}$ be their respective liftings to $\mathbb{R}$ beginning at 0.

Let $n = \tilde{f}(1)$, $m = \tilde{g}(1)$. Then $\Phi([f]) = n$, $\Phi([g]) = m$.

Let $\hat{g} = n + \tilde{g}$.
Since $p(n + x) = p(x)$ $\forall x \in \mathbb{R}$, the path $\hat{g}$ is a lifting of g beginning at n.

Note: $\tilde{f} * \hat{g}$ is the lifting of $f * g$ beginning at 0 and ending at $\hat{g}(1) = n + \tilde{g}(1) = n + m$.

Thus, $\Phi([f * g]) = \tilde{f} * \hat{g}(1) = n + m$
$= \Phi([f]) + \Phi([g])$.